arXiv:1303.6214v1 [math.AC] 25 Mar 2013

A NOTE ON THE SUBADDITIVITY PROBLEM FOR MAXIMAL SHIFTS IN FREE RESOLUTIONS

JUERGEN HERZOG AND HEMA SRINIVASAN

Abstract. We present some partial results regarding subadditivity of maximal shifts in finite graded free resolutions.

Let K be field, $S = K[x_1, \dots, x_n]$ the polynomial ring over K in the indeterminates $x_1, \dots, x_n$ and $I \subset S$ a graded ideal. Let $(\mathbb{F}, \partial)$ be a graded free S-resolution of $R = S/I$. Each free module $\mathbb{F}_a$ in the resolution is of the form $\mathbb{F}_a = \bigoplus_j S(-j)^{b_{aj}}$. We set

$$t_a(\mathbb{F}) = \max\{j\colon\ b_{aj} \neq 0\}.$$

In the case that $\mathbb{F}$ is the graded minimal free resolution of I we write $t_a(I)$ instead of $t_a(\mathbb{F})$.

We say $\mathbb{F}$ satisfies the *subadditivity condition*, if $t_{a+b}(\mathbb{F}) \leq t_a(\mathbb{F}) + t_b(\mathbb{F})$.

Remark 1. The Taylor complex and the Scarf complex satisfy the subadditivity condition. Indeed, both complexes are cellular resolutions supported on a simplicial complex. From this fact the assertion follows immediately.

The minimal resolution of a graded algebra S/I does not always satisfy the subadditivity condition as pointed out in [1]. Additional assumptions on the ideal I are required. Somewhat weaker inequalities can be shown in certain ranges of a and b, and in particular the inequality $t_{a+1}(I) \leq t_a(I) + t_1(I)$ if $R = S/I$ is Koszul and $a \leq \operatorname{height} I$, see [1, Theorem 4.1]. Another case of interest for which the subadditivity condition holds is when $\dim S/I \leq 1$ and $a + b = n$ as shown in [3, Theorem 4.1] by David Eisenbud, Craig Huneke and Bernd Ulrich. No counterexample is known for monomial ideals,

For a general graded ideal I we have the following result.

Proposition 2. *Let $I \subset S$ be a graded ideal, $\mathbb{F}$ the graded minimal free resolution of S/I. Suppose there exists a homogeneous basis $f_1, \dots, f_r$ of F_a such that*

$$\partial(\mathbb{F}_{a+1}) \subset \bigoplus_{i=1}^{r-1} S f_i.$$

Then $\deg f_r \leq t_{a-1} + t_1$.

2000 *Mathematics Subject Classification.* 13A02, 13D02.

Key words and phrases. Graded free resolutions, monomial ideals.

The paper was written while the authors were visiting MSRI at Berkeley. They wish to acknowledge the support, the hospitality and the inspiring atmosphere of this institution.

Proof. We denote by $(\mathbb{F}^*, \partial^*)$ the complex $\operatorname{Hom}_S(\mathbb{F}, S)$ which is dual to $\mathbb{F}$. For any basis $h_1, \ldots, h_l$ of $\mathbb{F}_b$ we denote by h_i^* the basis element of $\mathbb{F}_b^*$ with $h_i^*(h_j) = 1$ if $j = i$ and $h_i^*(h_j) = 0$, otherwise. Then $h_1^*, \ldots, h_l^*$ is a basis of $\mathbb{F}_b^*$, the so-called dual basis of $h_1, \ldots, h_l$.

Our assumption implies that $\partial^*(f_r^*) = 0$. This implies that f_r^* is a generator of $H^a(\mathbb{F}^*) = \operatorname{Ext}_S^a(S/I, S)$, and hence $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, since $\operatorname{Ext}^a(S/I, S)$ is an S/I-module. On the other hand, if $g_1, \ldots, g_m$ is a basis of $\mathbb{F}_{a-1}$ and $\partial(f_r) = c_1 g_1 + \cdots + c_n g_m$, then $\partial^*(g_i^*) = c_i f^* + m_i$ where each m_i is a linear combination of the remaining basis elements of $\mathbb{F}_a^*$. Let $c \in I$ be a generator of maximal degree. Then by definition, $\deg c = t_1(I)$. Since $If_r^* = 0$ in $H^a(\mathbb{F}^*)$, there exist homogeneous elements $s_i \in S$ such that $cf_r^* = \sum_{i=1}^m s_i(c_i f_r^* + m_i)$. This is only possible if $t_1(I) = \deg c_i + \deg s_i$ for some i. In particular, $\deg c_i \leq t_1(I)$. It follows that $\deg f_r = \deg c_i + \deg g_i \leq t_1(I) + t_{a-1}(I)$, as desired. □

In [6, Theorem 4.4] Jason McCullough shows that $t_p(I) \leq \max_a\{t_a(I) + t_{p-a}(I)\}$ where $p = \operatorname{proj}\dim S/I$. As an immediate consequence of Proposition 2 we obtain the following improvement of McCullough's inequality:

Corollary 3. *Let $I \subset S$ be a graded ideal of projective dimension p. Then*

$$t_p(I) \leq t_{p-1}(I) + t_1(I).$$

For monomial ideals one even has

Corollary 4. *Let I be a monomial ideal. Then $t_a(I) \leq t_{a-1}(I) + t_1(I)$ for all $a \geq 1$.*

For the proof of this and the following results we will use the restriction lemma as given in [5, Lemma 4.4]: let I be a monomial ideal with multigraded (minimal) free resolution $\mathbb{F}$ and let $\alpha \in \mathbb{N}^n$. Then the restricted complex $\mathbb{F}^{\leq \alpha}$ which is the subcomplex of $\mathbb{F}$ for which $(\mathbb{F}^{\leq \alpha})_i$ is spanned by those basis elements of $\mathbb{F}_i$ whose multidegree is componentwise less than or equal to α, is a (minimal) multigraded free resolution of the monomial ideal $I^{\leq \alpha}$ which is generated by all monomials $\mathbf{x}^{\mathbf{b}} \in I$ with $\mathbf{b} \leq \alpha$, componentwise.

Proof of Corollary 4. Let $\mathbb{F}$ the minimal multigraded free S-resolution of S/I, and let $f \in F_a$ be a homogeneous element of multidegree $\alpha \in \mathbb{N}^n$ whose total degree is $t_a(I)$. We apply the restriction lemma and consider the restricted complex $\mathbb{F}^{\leq \alpha}$. Let $f_1, \ldots, f_r$ be a homogenous basis of $(\mathbb{F}^{\leq \alpha})_a$ with $f_r = f$. Since there is no basis element of $(\mathbb{F}^{\leq \alpha})_{a+1}$ of a multidegree which is coefficient bigger than α, and since the resolution $\mathbb{F}^{\leq \alpha}$ is minimal, it follows that $\partial((\mathbb{F}^{\leq \alpha})_{a+1}) \subset \bigoplus_{i=1}^{r-1} S f_i$. Thus we may apply Proposition 2 and deduce that $t_a(I^{\leq \alpha}) \leq t_{a-1}(I^{\leq \alpha}) + t_1(I^{\leq \alpha})$. Since $t_a(I) = t_a(I^{\leq \alpha})$, $t_{a-1}(I^{\leq \alpha}) \leq t_{a-1}(I)$ and $t_1(I^{\leq \alpha}) \leq t_1(I)$, the assertion follows. □

The preceding corollary generalizes [2, Corollary 1.9] of Fernández-Ramos and Philippe Gimenez, who showed that $t_a \leq t_{a-1} + 2$ for any monomial ideal generated in degree 2.

Let $I \subset S$ be a monomial ideal, and $\alpha, \beta \in \mathbb{N}^n$ be two integer vectors. We say that (α, β) is a *covering pair* for I, if

$$I = I^{\leq \alpha} + I^{\leq \beta}.$$

Theorem 5. *Let (α, β) be a covering pair for the monomial ideal I, and suppose that $p = \operatorname{proj\,dim} S/I^{\leq \alpha}$ and $q = \operatorname{proj\,dim} S/I^{\leq \beta}$. Then $\operatorname{proj\,dim} S/I \leq p + q$, and for all integers $a \leq \operatorname{proj\,dim} S/I$ we have*

$$t_a(I) \leq \max\{t_i(I) + t_j(I) \colon\ i + j = a,\ i \leq p,\ j \leq q\}.$$

Proof. We consider the complex $\mathbb{G} = \mathbb{F}^{\leq \alpha} * \mathbb{F}^{\leq \beta}$ defined in [4]. Then $\mathbb{G}$ is a multigraded free resolution of $I^{\leq \alpha} + I^{\leq \beta}$ of length $p + q$, and hence a multigraded free resolution of I. In particular, it follows that $\operatorname{proj\,dim} S/I \leq p + q$.

By construction,

$$\mathbb{G}_a = \bigoplus_{i+j=a} (\mathbb{F}^{\leq \alpha})_i * (\mathbb{F}^{\leq \beta})_j,$$

where each direct summand $(\mathbb{F}^{\leq \alpha})_i * (\mathbb{F}^{\leq \beta})_j$ is a free multigraded S-module. If $f_1, \ldots, f_s$ is a multihomogeneous basis of $(\mathbb{F}^{\leq \alpha})_i$ and $g_1, \ldots, g_r$ a multihomogeneous basis of $(\mathbb{F}^{\leq \beta})_j$, then the symbols $f_k * g_l$ with $k = 1, \ldots, s$ and $l = 1, \ldots, r$ establish a multihomogeneous basis of $(\mathbb{F}^{\leq \alpha})_i * (\mathbb{F}^{\leq \beta})_j$, and if σ_k is the multidegree of f_k and τ_l is the multidegree of g_l, then $\sigma_k \vee \tau_l$ is the multidegree of $f_k * g_l$, where for two integer vectors $\gamma, \delta \in \mathbb{N}^n$ we denote by $\gamma \vee \delta$ the integer vector which is obtained from γ and δ by taking componentwise the maximum. It follows that the element of maximal (total) degree in $(\mathbb{F}^{\leq \alpha})_i * (\mathbb{F}^{\leq \beta})_j$ has degree less than or equal to $t_i(\mathbb{F}^{\leq \alpha}) + t_j(\mathbb{F}^{\leq \beta})$. Consequently we obtain

$$\begin{aligned} t_a(I) &= t_a(\mathbb{F}) \leq t_a(\mathbb{G}) \leq \max\{t_i(\mathbb{F}^{\leq \alpha}) + t_j(\mathbb{F}^{\leq \beta}) \colon\ i + j = a\ , i \leq p\ , j \leq q\} \\ &\leq \max\{t_i(I) + t_j(I) \colon\ i + j = a\ , i \leq p\ , j \leq q\}. \end{aligned}$$

□

The following example illustrates that Theorem 5 leads to inequalities which are not implied by Corollary 3.

Example 6. Let $S = k[x, y, z, u, v, w, a]$ and

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2, x^5, y^5, z^5, u^5, w^5, v^6, a^6) \subset S.$$

We choose $\alpha = (5, 5, 5, 5, 0, 0, 0)$ and $\beta = (3, 3, 2, 2, 6, 5, 6)$. Then

$I^{\leq \alpha} = (x^5, y^5, z^5, u^5, x^3y^3z^2, u^2y^2z^3)$, $I^{\leq \beta} = (w^5, v^6, a^6, x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2)$.

Here, $p = 4$, $q = 5$ and $\operatorname{proj\,dim} S/I = 7$. Thus by Theorem 5,

$$t_7(I) \leq \max\{t_2(I) + t_5(I), t_3(I) + t_4(I)\}.$$

Corollary 7. *Let $s = p + q - a$. Then with the notation and assumptions of Theorem 5 we have*

$$t_a(I) \leq \max\{t_i(I) + t_{a-i}(I) \colon\ p - s \leq i \leq p\}.$$

A a special case of this corollary one obtains

Corollary 8. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal with $\dim S/I = 0$ which is minimally generated by $m \leq 2n-6$ monomials, and let a be an integer with $(m+4)/2 \leq a \leq n$. Then for all $p = m-a+2, \ldots, a-2$,*

$$t_a(I) \leq \min\{t_1(I) + t_{a-1}(I), \max\{t_i(I) + t_{a-i}(I) \colon\ p-(m-a) \leq i \leq \min\{p, a/2\}\}\}.$$

Proof. Due to Corollary 3 we only need to show that

$$t_a(I) \leq \max\{t_i(I) + t_{a-i}(I) \colon\ p+a-m \leq i \leq \min\{p, a/2\}\}.$$

Since $\dim S/I = 0$, it follows that among the minimal set of generators $G(I)$ of I are the pure powers $x_1^{a_1}, \ldots, x_n^{a_n}$ for suitable $a_i > 0$. We let $\alpha = (a_1, \ldots, a_p, 0, \ldots, 0)$. Then $I^{\leq\alpha}$ has all its generators in $K[x_1, \ldots, x_p]$ so that $\operatorname{proj}\dim S/I = p$. Let J be the ideal which is generated by the set of monomials $G(I) \setminus \{x_1^{a_1}, \ldots, x_p^{a_p}\}$, and let x^β be the least common multiple of the generators of J. Then $J = I^{\leq\beta}$ and (α, β) is a covering pair for I. Since J is generated by $m-p$ elements it follows that $q = \operatorname{proj}\dim S/J \leq m-p$. Hence the desired inequality follows from Corollary 7. The conditions on the integers a, m and p only make sure that $i \geq 2$ and $a-i \geq 2$ for all i with $p+a-m \leq i \leq p$, and that $m-a+2 \leq a-2$. $\square$

The bound in Corollary 8 is a partial improvement of the results in [3] and [6] since the bound is also valid for certain $a < n$. For $a = n$, it is weaker than the one in [3] for zero dimensional rings and is stronger than the one in [6]. For example, if $n = 7$ and $m = 8$ one has $t_6 \leq t_1 + t_2 + t_3$, and if $6 \leq n \leq 20$ and $m \leq 2n-6$, then one has $t_7 \leq t_1 + t_2 + t_4$.

Remark 9. With the same methods as applied in the proof of Theorem 5 one can show the following statement: let $I \subset S$ be a monomial ideal with graded minimal free resolution $\mathbb{F}$, and $f_i \in F_{a_i}$ multihomogeneous basis elements of multidegree α_i for $i = 1, \ldots, r$. Assume that $I = \sum_{i=1}^r I^{\leq\alpha_i}$. Then

$$t_{a_1+a_2+\cdots+a_r}(I) \leq t_{a_1}(I) + t_{a_2}(I) + \cdots + t_{a_r}(I).$$

To satisfy the condition $I = \sum_{i=1}^r I^{\leq\alpha_i}$ requires in general that either r is big enough or that the α_i are large enough (with respect to the partial order given by componentwise comparison). Here is an example with $r = 2$ to which Remark 9 applies: let

$$I = (x^2w^2v^2, a^2x^3y^2u^2w^2, a^2z^2u^2, u^2y^2z^3, x^3y^2z^2) \subset k[x, y, z, w, u, v, a]$$

The Betti numbers of R/I are $1, 5, 8, 5, 1$. Even though the Betti sequence is symmmetric, the ideal I is not Gorenstein, since it is of height 2 and projective dimension 4. The two multidegrees in F_2 which form a covering pair for I are $(3, 2, 2, 2, 2, 0, 2)$ and $(2, 2, 3, 2, 2, 2, 0)$. In this example we have $t_1 = 11, t_2 = 13, t_3 = 15, t_4 = 16$ and we clearly have $t_i \leq t_2 + t_2$.

References

[1] Luchezar L. Avramov, Aldo Conca, and Srikanth B. Iyengar, Subadditivity of Syzygies of Koszul algebras, Preprint 2012.

[2] Oscar Fernández-Ramos, and Philippe Gimenez, Regularity 3 in edge ideals associated to bipartite graphs, Preprint 2011, arXiv:1108.3203.

[3] David Eisenbud, Craig Huneke, and Bernd Ulrich, The regularity of Tor and graded Betti numbers, Amer. J. Math. **128** (2006), no. 3, 573–605.

[4] Jürgen Herzog, A generalization of the Taylor complex construction, Comm.Alg. **35** (2007), 1747–1756.

[5] Jürgen Herzog, Takayuki Hibi and Xinxian Zheng, Monomial ideals whose powers have a linear resolution. Math. Scand. **95** (2004), no. 1, 23–32.

[6] Jason McCullough, A polynomial bound on the regularity of an ideal in terms of half the syzygies, Prerint 2012, arXiv:1112.005

Jürgen Herzog, Fachbereich Mathematik, Universität Duisburg-Essen, Campus Essen, 45117 Essen, Germany

E-mail address: juergen.herzog@uni-essen.de

Hema Srinivasan, Mathematics Department, University of Missouri, Columbia, MO 65211, USA

E-mail address: SrinivasanH@math.missouri.edu